Variational approach to the QCD wave functional: Dynamical mass generation and confinement

Ian I. Kogan*
Physics Department, Princeton University, Princeton, New Jersey 08544

Alex Kovner†
Theory Division, T-8, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
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We perform a variational calculation in the SU(N) Yang-Mills theory in $3+1$ dimensions. Our trial variational states are explicitly gauge invariant, and reduce to simple Gaussian states in the zero coupling limit. Our main result is that the energy is minimized for the value of the variational parameter away from the perturbative value. The best variational state is therefore characterized by a dynamically generated mass scale M. This scale is related to the perturbative scale $\Lambda_{\rm QCD}$ by the relation $\alpha_{\rm QCD}(M) = (\pi/4N)$. Taking the one-loop QCD β function and $\Lambda_{\rm QCD} = 150$ MeV we find (for $N = 3$) the vacuum condensate $(\alpha/\pi)\langle F^2\rangle = 0.008$ GeV^4.

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I. INTRODUCTION

Understanding of low energy phenomena in QCD, such as confinement and chiral symmetry breaking or, in more general terms, the strong coupling problem and a ground state structure in an asymptotically free non-Abelian gauge theory, is, without doubt, one of the main (if not the main) problems in modern quantum field theory. In spite of years of attempts to answer this question, we are still far from complete satisfaction, although a lot of interesting and promising ideas were suggested during the first 20 years of QCD [1].

Considerable progress has been made in this direction during recent years using the numerical approach of lattice gauge theory [2]. The lattice gauge theory calculations are, however, still incomplete. Apart from that, they sometimes leave behind an unpleasant aftertaste (although this is, of course, a very subjective matter) that one obtains numerical results without gaining a real understanding of the underlying physics. To our minds, the understanding of these issues in the framework of an analytical approach would be invaluable. An analytic method that is capable of solving the low energy sector of QCD starting from first principles would also, presumably, teach us a lot about other strongly interacting theories such as technicolor.

Unfortunately, the arsenal of nonperturbative methods to tackle strongly interacting continuum theories is very limited, to say the least. Methods that perform very well in simple quantum-mechanical problems are much more difficult to use in quantum field theory (QFT). This is true, for example, for a variational approach. In quantum mechanics it is usually enough to know a few simple qualitative features in order to set up a variational ansatz which gives pretty accurate results, not only for the energy of a ground state, but also for various other vacuum expectation values (VEV's). In QFT one is immediately faced with several difficult problems when trying to apply this method, as discussed insightfully by Feynman [3].

First, there is the problem of calculability. That is, even if one had a very good guess at the form of the vacuum wave functional (or, for that matter, even knew its exact form), one would still have to evaluate expectation values of various operators in this state. In a field theory in d spatial dimensions, this involves performing a d-dimensional path integration, a problem, in itself very complicated and, in general, not manageable. This problem is especially severe in non-Abelian gauge theories, where gauge invariance poses strong restrictions on admissible trial wave functionals (WF's). In this case it becomes very difficult to find a set of WF's which are both gauge invariant and amenable to analytic calculation.

Another serious problem is the problem of "ultraviolet modes." This means the following. In a variational calculation of the kind we have in mind, one is mostly interested in information about the low momentum modes. However, the VEV of the energy (and all other intensive quantities) is dominated entirely by contributions of high momentum fluctuations, for a simple reason, that there are infinitely more UV modes than modes with low momentum. Therefore, even if one has a very good idea how the WF's at low momenta should look, if the UV part of the trial state is even slightly incorrect the minimization of energy may lead to absurd results. Because of the interaction between the high and low momentum modes, the IR variational parameters will in general be driven to values which minimize the interaction energy

*On leave of absence from ITEP, B. Cheremyshkinskaya 25, Moscow 117259, Russia. Present address: Department of Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK.
†Present address: Physics Department, University of Minnesota, 116 Church Street S.E., Minneapolis, MN 55455.

and have nothing to do with the dynamics of the low momentum modes themselves.

Even though over the years many attempts at variational calculations in QCD have been made [4], these two problems invariably made their presence felt, and at this point one really cannot point to any successful variational calculation in a non-Abelian gauge theory. Our feeling is, however, that these obstacles are not necessarily insurmountable and that this direction is still far from being exhausted and deserves further development.

In this paper we present a variational calculation of the vacuum WF in a pure SU(N) Yang-Mills theory, which, at least partially, is free from the problems mentioned earlier. We use wave functionals that are explicitly gauge invariant. The correct UV behavior is built into our ansatz. In the case at hand, this can be done due to the asymptotic freedom of the models considered. We are able to calculate VEV's of local operators in our trial states in a reasonable approximation, combining the renormalization group and the mean field techniques.

Our main result is that the energy is minimized at the value of the variational parameter away from the perturbative vacuum state. This leads to a dynamical generation of scale in the vacuum WF. The value of the vacuum condensate $\alpha/\pi\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle$ in this state turns out to be equal to 0.008 GeV^4 for $\Lambda_{\rm QCD} = 150$ MeV. Even though this result should be taken only as an order of magnitude estimate (as a result of approximations made), it is pleasing to see a number so close to the phenomenologically known 0.012 GeV^4 [5] emerge from this simple calculation.

The paper is organized as follows. In Sec. II we set up the variational calculation and discuss in some detail our variational ansatz. In Sec. III we discuss the approximation scheme for calculation of VEV's in the trial WF. In Sec. IV the minimization of energy and calculation of $\langle F^2\rangle$ are performed. Some elements of the calculation of the Wilson loop and an area law are discussed in Sec. V. Finally, Sec VI contains a discussion of our results and outlines directions for future work.

II. VARIATIONAL TRIAL STATE AND THE GAUGE INVARIANCE

As mentioned in the previous section, an immediate question one is faced with when picking a possible variational state is calculability. One should be able to calculate the averages of local operators in this state:

$$\langle O\rangle = \int D\phi\Psi^*[\phi]O\Psi[\phi]\ . \tag{2.1}$$

A calculation of this kind, obviously, is tantamount to evaluation of a Euclidean path integral with the square of the WF playing the role of the partition function. One should therefore be able to solve exactly a d-dimensional field theory with the action

$$S[\phi] = -\ln\Psi[\phi]\Psi[\phi]\ . \tag{2.2}$$

Since in dimension $d > 1$ the only theories one can solve exactly are free field theories, the requirement of calculability almost unavoidably restricts the possible form of the WF to a Gaussian (or as it is sometimes called squeezed) state:

$$\Psi[\phi] = \exp\left\{-\frac{1}{2}\int d^3x\, d^3y[\phi(x)-\zeta(x)]G^{-1}(x,y)[\phi(y)-\zeta(y)]\right\}\ , \tag{2.3}$$

with $\zeta(x)$ and $G(x,y)$ being c-number functions. The requirement of translational invariance usually gives further restrictions: $\zeta(x) = \text{const}$, $G(x,y) = G(x-y)$.

The restriction to a Gaussian WF is of course a severe one. However, one can still hope that in some cases the simple Gaussian form can capture the most important nonperturbative characteristics of the true vacuum. Indeed, the Gaussian variational approximation has been used successfully in self-interacting scalar theories, where it is known to be exact in the limit of large number of fields. Perhaps the most celebrated use of these trial states is the BCS calculation of the superconducting ground state [6], where for most of the interesting quantities its accuracy is of order 10–20 %.

The reason the approximation works well in these theories is that in both cases a single condensate dominates the nonperturbative physics and the Gaussian ansatz is wide enough to accommodate this most important condensate. From this point of view, it would seem then that it is perfectly reasonable to try a similar variational ansatz in the Yang-Mills theory. After all, it is strongly suggested by QCD sum rules [5] that the pure glue sector is dominated by one nonperturbative condensate $\langle F^2\rangle$. We also know that the VEV of the field strength itself $\langle F\rangle$ vanishes, since it is not a gauge-invariant operator. A state of the form (2.3) with $\zeta = 0$ would indeed give zero classical fields, but nonzero quadratic condensates.

There is, however, one obvious difficulty with this idea. It is very easy to see that in a non-Abelian theory it is impossible to write down a Gaussian WF which satisfies the constraint of gauge invariance. The SU(N) gauge theory is described by a Hamiltonian

$$H = \int d^3x\left[\tfrac{1}{2}E^{a2}_i + \tfrac{1}{2}B^{a2}_i\right]\ , \tag{2.4}$$

where

$$E^a_i(x) = i\frac{\delta}{\delta A^a_i(x)}\ ,$$
$$B^a_i(x) = \tfrac{1}{2}\epsilon_{ijk}\{\partial_j A^a_k(x) - \partial_k A^a_j(x) + gf^{abc}A^b_j(x)A^c_k(x)\}\ , \tag{2.5}$$

and all physical states must satisfy the constraint of gauge invariance:

$$G^a(x)\Psi[A] = \left[\partial_i E_i^a(x) - g f^{abc} A_i^b(x) E_i^c(x)\right]\Psi[A] = 0 \ . \tag{2.6}$$

Under a gauge transformation U [generated by $G^a(x)$], the vector potential transforms as

$$A_i^a(x) \to A_i^{Ua}(x) = S^{ab}(x)A_i^b(x) + \lambda_i^a(x) \ , \tag{2.7}$$

where

$$S^{ab}(x) = \tfrac{1}{2}\mathrm{tr}(\tau^a U^\dagger \tau^b U), \quad \lambda_i^a(x) = \frac{i}{g}\mathrm{tr}\left(\tau^a U^\dagger \partial_i U\right) \ , \tag{2.8}$$

and τ^a are traceless Hermitian $N \times N$ matrices satisfying $\mathrm{tr}(\tau^a\tau^b) = 2\delta^{ab}$. A Gaussian wave functional

$$\Psi[A_i^a] = \exp\left\{-\tfrac{1}{2}\int d^3x\, d^3y [A_i^a(x) - \zeta_i^a(x)] \times (G^{-1})_{ij}^{ab}(x,y)[A_j^b(y) - \zeta_j^b(y)]\right\} \tag{2.9}$$

transforms under the gauge transformation as

$$\Psi[A_i^a] \to \Psi\left[(A^U)_i^a\right] \ . \tag{2.10}$$

In the Abelian case it is enough to take $\partial_i G_{ij}^{-1} = 0$ to satisfy the constraint of gauge invariance. In the non-Abelian case, however, because of the homogeneous piece in the gauge transformation (2.7), no gauge-invariant Gaussian WF's exist.

One possible strategy is to disregard this fact [7] and hope that one does not lose much by minimizing the energy in the whole Hilbert space, which also includes unphysical states. This is, however, very risky. The sticking point is that the Hamiltonian of the theory is unique *only* on physical states. One can add to Eq. (2.4) an arbitrary operator multiplied by one of the generators of the gauge group without changing the energies of the physical states, but reshuffling the rest of the spectrum beyond recognition. In this way the gap between the physical vacuum and some of the unphysical states can be made very small. In fact, the large Hilbert space can even contain states which have energies lower than the physical vacuum. Since we are working with a particular Hamiltonian, it is not clear *a priori* that this is not the case. Therefore minimizing the energy on the whole space may lead to huge admixtures of unphysical states in the "best variational state," making the results of such a procedure meaningless. Of course, one could be lucky and with the particular choice of the Hamiltonian (2.4) all unphysical states may have large energies, but there is no way to know it without a separate investigation of this question.

We, at any rate, will restrict our attention to gauge-invariant states only. It is clear then that the Gaussian ansatz must be modified. Several modifications were considered in previous work. One obvious possibility is to restrict classical fields to zero and insert adjoint Wilson lines in the exponential [8], so that

$$[A_i^a(x) - \zeta_i^a(x)](G^{-1})_{ij}^{ab}(x,y)\left[A_j^b(y) - \zeta_j^b(y)\right] \to B_i^a(x)G_{ij}^{-1}(x-y)B_j^b(y)W^{ab}(C) \ , \tag{2.11}$$

where $W(C) = P\exp(ig\int_C dl_i F^a A_i^a)$ and F^a are the generators of SU(N) in the adjoint representation. This form, however, makes it practically impossible to perform explicit calculations, except in the weak coupling limit. Another proposed modification is to multiply the Gaussian by a finite-order polynomial in the fields. In that way gauge invariance can be maintained to a finite order in the coupling constant [9]. Then, however, it is again not quite clear to which extent the calculation is nonperturbative.

Instead, we will take a straightforward approach and simply project the Gaussian WF onto a gauge-invariant sector. In this paper we also restrict ourselves to the case of zero classical fields ($\zeta = 0$). Our variational ansatz is therefore

$$\Psi[A_i^a] = \int DU(x)\exp\left\{-\tfrac{1}{2}\int d^3x\, d^3y A_i^{Ua}(x)G_{ij}^{-1ab}(x-y)A_j^{Ub}(y)\right\} \ , \tag{2.12}$$

with A_i^{Ua} defined in (2.7) and the integration is performed over the space of special unitary matrices with the SU(N)-group-invariant measure.

Before attempting a calculation with this expression, we will impose several restrictions on the form of G, which will lead to considerable simplifications. First, we will only consider matrices G of the form

$$G_{ij}^{ab}(x-y) = \delta^{ab}\delta_{ij}G(x-y) \ . \tag{2.13}$$

This form is certainly the right one in the perturbative regime. In the leading order in perturbation theory, the non-Abelian character of the gauge group is not important, and the integration in Eq. (2.12) is basically over the $\mathrm{U}(1)^{N^2-1}$ group. The δ^{ab} structure is then obvious — there is a complete democracy between different components of the vector potential. The δ_{ij} structure arises in the following way. If not for the integration over the group, G_{ij}^{-1} would be precisely the (equal time) propagator of the electric field. However, because of the integration over the group, the actual propagator is the transverse part of G^{-1}. It is easy to check that the longitudinal part $\partial_i G_{ij}^{-1}$ drops out of all physical quantities. At the perturbative level, therefore, one can take $G_{ij} \sim \delta_{ij}$ without any loss of generality. We will adopt this form

of the matrix G also in our variational calculation.

We can use additional perturbative information to restrict the form of G even further. The theory of interest is asymptotically free. This means that the short distance asymptotics of correlation functions must be the same as in the perturbation theory. Since G^{-1} in perturbation theory is directly related to correlation functions of gauge-invariant quantities (e.g., E^2), we conclude

$$G^{-1}(x) \to \frac{1}{x^4}, \quad x \to 0 . \tag{2.14}$$

Finally, we expect the theory nonperturbatively to have a gap. In other words, the correlation functions should decay to zero at some distance scale:

$$G(x) \sim 0, \quad x > \frac{1}{M} . \tag{2.15}$$

We will build this into our variational ansatz in the simplest possible way. We will take M to be our only variational parameter. This can be done by choosing for $G(x)$ a particular form that has the UV and IR asymptotics given by (2.14) and (2.15), such as for example, a massive scalar propagator with mass M. We find another parametrization slightly more convenient. The form that will be used throughout this calculation has the Fourier transform

$$G^{-1}(k) = \begin{cases} \sqrt{k^2} & \text{if } k^2 > M^2 , \\ M & \text{if } k^2 < M^2 . \end{cases} \tag{2.16}$$

We have checked that using a massive propagator instead practically does not change the results. Equation (2.12) together with Eqs. (2.13) and (2.16) defines our variational ansatz. We now have to calculate the energy expectation value in these states and minimize it with respect to the only variational parameter left — the scale M. Note that the perturbative vacuum is included in this set of states and corresponds to $M = 0$. A nonzero result for M would therefore mean a nonperturbative dynamical scale generation in the Yang-Mills vacuum. In the next section we will explain the approximation scheme we use to calculate expectation values in the trial state.

III. EFFECTIVE σ MODEL AND THE RENORMALIZATION GROUP

The question now is how to calculate the expectation values in the state of the form (2.12):

$$\langle O \rangle = \frac{1}{Z} \int DU' DU'' \langle O \rangle_A ,$$

$$\langle O \rangle_A = \int DA \exp\left(-\tfrac{1}{2} \int dx\, dy A_i^{U'a}(x) G^{-1}(x-y) A_i^{U'a}(y) \right) O \exp\left(-\tfrac{1}{2} \int dx'\, dy' A_j^{U''b}(x') G^{-1}(x'-y') A_j^{U''b}(y') \right) , \tag{3.1}$$

where Z is the norm of the trial state. Two simplifications are immediately obvious. First, since we will only be considering gauge-invariant operators O, one of the group integrations is redundant. Performing the change of variables $A \to A^{U''}$ (and remembering that both integration measures DU and DA are group invariant), we obtain [omitting the volume of SU(N) factor $\int D(U'U'')$]

$$\langle O \rangle = \frac{1}{Z} \int DU \langle O \rangle_A ,$$

$$\langle O \rangle_A = \int DA \exp\left(-\tfrac{1}{2} \int dx\, dy A_i^{Ua}(x) G^{-1}(x-y) A_i^{Ua}(y) \right) O \exp\left(-\tfrac{1}{2} \int dx'\, dy' A_j^{b}(x') G^{-1}(x'-y') A_j^{b}(y') \right) , \tag{3.2}$$

where we have defined $U = U'U''^{\dagger}$. Also, since the gauge transform of a vector potential is a linear function of A [Eq. (2.7)], for fixed $U(x)$ this is a Gaussian integration and can therefore be performed explicitly for any reasonable operator O. We are left then only with a path integral over one group variable $U(x)$. But this a tough one![1]

Let us consider first the normalization factor Z. After integrating over the vector potential, we obtain

$$Z = \int DU \exp\{-\Gamma[U]\} , \tag{3.3}$$

[1]The functional integral over U in what follows will be treated approximately. However, we want to stress here that approximations at this point do not introduce any non-gauge-invariant effects into our calculation. The important point is that the approximations are made only after integrating over $U'U''$ in Eq. (3.1). It is this integration that wipes out contributions of any gauge nonsinglet piece in O. Integration over U can therefore be approximated without harming the exact gauge invariance of our calculation.

with an action

$$\Gamma[U] = \tfrac{1}{2}\mathrm{Tr}\ln\mathcal{M} + \tfrac{1}{2}\lambda[G + SGS^T]^{-1}\lambda \ , \tag{3.4}$$

where multiplication is understood as the matrix multiplication with indices: color a, space i, and position (the values of space coordinates) x, i.e.,

$$(AB)^{ac}_{ik}(x,z) = \int d^3y\, A^{ab}_{ij}(x,y)B^{bc}_{jk}(y,z), \quad \lambda O\lambda = \int d^3x\, d^3y \lambda^a_i(x)O^{ij}_{ab}(x-y)\lambda^b_j(y) \ . \tag{3.5}$$

The trace Tr is understood as a trace over all three types of indices. In Eq. (3.4) we have defined

$$S^{ab}_{ij}(x,y) = S^{ab}(x)\delta_{ij}\delta(x-y), \quad \mathcal{M}^{ab}_{ij}(x,y) = [S^{Tac}(x)S^{cb}(y) + \delta^{ab}]G^{-1}(x-y)\delta_{ij} \ , \tag{3.6}$$

where $S^{ab}(x) = \frac{1}{2}\mathrm{tr}(\tau^a U^\dagger \tau^b U)$ and $\lambda^a_i(x) = \frac{i}{g}\mathrm{tr}(\tau^a U^\dagger \partial_i U)$ were defined in (2.8) and tr is a trace over colour indices only. Using the completeness condition for SU(N),

$$\tau^a_{ij}\tau^a_{kl} = 2\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \ , \tag{3.7}$$

one can see that S^{ab} is an orthogonal matrix:

$$\begin{aligned} S^{ab}S^{cb} &= \tfrac{1}{4}\tau^b_{ij}\tau^b_{kl}(U\tau^a U^\dagger)_{ji}(U\tau^c U^\dagger)_{kl} \\ &= \tfrac{1}{2}\mathrm{tr}(\tau^a\tau^b) = \delta^{ab} \ , \end{aligned} \tag{3.8}$$

where we used that tr $(U\tau^c U^\dagger) = \mathrm{tr}\tau^c = 0$.

We have written action (3.4) in a form which suggests a convenient way of thinking about the problem. The path integral (3.3) defines a partition function of a nonlinear σ model with the target space SU(N)/Z_N in three-dimensional Euclidean space. The fact that the target space is SU(N)/Z_N rather than SU(N) follows from the observation that the action (3.4) is invariant under local transformations belonging to the center of SU(N). This can be trivially traced back to invariance of A^a_i under gauge transformations that belong to the center of the gauge group.

We note that the quantity $U(x)$ has a well-defined gauge-invariant meaning, and it is *not* itself a matrix of a gauge transformation. A contribution of a given $U(x)$ to the partition function (3.3) and to other expectation values corresponds to the contribution to the same quantity from the off-diagonal matrix element between the initial Gaussian and the Gaussian gauge rotated by $U(x)$. Therefore, if matrices $U(x)$ which are far from unity give a significant contribution to the partition function, it means that the off-diagonal contribution is large and therefore that the simpleminded non-gauge-invariant Gaussian approximation (which neglects the off diagonal elements) misses important physics.

The action of this σ model is rather complicated. It is a nonlocal and a nonpolynomial functional of $U(x)$. There are, however, two observations that will help us devise an approximation scheme to deal with the problem. First, remembering that the bare coupling constant of the Yang Mills theory is small, let us see how does it enter the σ model action. It is easy to see that the only place it enters is the second term in the action (3.4), because $\lambda^a_i(x)$ has an explicit factor $1/g$. Moreover, it enters in the same way as a coupling constant in a standard σ model action. We can therefore easily set up a perturbation theory in our σ model. With the standard parametrization

$$U(x) = \exp\left\{i\frac{g}{2}\phi^a\tau^a\right\} \ , \tag{3.9}$$

one gets $\lambda^a_i(x) = -\partial_i\phi^a(x) + O(g)$, $S^{ab}(x) = \delta^{ab} + O(g)$, and the leading order term in the action becomes

$$\tfrac{1}{16}\int d^3x\, d^3y \partial_i\phi^a(x)G^{-1}(x-y)\partial_i\phi^a(y) \ . \tag{3.10}$$

This is just a free theory, except that the propagator is nonstandard, and at large momenta its Fourier transform behaves like

$$D(k) \sim G(k)\frac{1}{k^2} \sim \frac{1}{|k|^3} \ . \tag{3.11}$$

Nevertheless, the perturbation theory is straightforward. Indeed, it is easy to see that in this σ model perturbation theory the coupling constant renormalizes logarithmically. The first order diagram that contributes to the coupling constant renormalization is the tadpole. In a σ model with a standard kinetic term, this diagram diverges linearly as $\int d^3k/k^2$, a sign of perturbative nonrenormalizability. In our model, though, because of the nonstandard form of the kinetic term (3.11), the diagram diverges only logarithmically as $\int d^3k/k^3$. The form of the β function therefore is very similar to the β function in ordinary QCD perturbation theory. In this paper we assume that to one loop the two β functions indeed coincide. The explicit calculation in the framework of the σ model will be presented elsewhere [10]. The perturbation theory, therefore, becomes worse and worse as we go to lower momenta and at some point becomes inapplicable.

Now, however, let us look at the other side of the coin. Let us see how does the action look like for the matrices $U(x)$, which are slowly varying in space. Because of the short range of $G(x)$, clearly for $U(x)$, which contain only momenta lower than the variational scale M, the action is local. In fact, with our ansatz (2.16) it becomes the standard action

$$\Gamma_L[U] = \frac{M}{2g^2}\mathrm{tr}\int d^3x\, \partial_i U^\dagger(x)\partial_i U(x) + \cdots \ , \tag{3.12}$$

where we omit the higher order in g terms. We have used the completeness condition (3.7) and the fact that tr $(U^\dagger \partial_i U) = 0$ to rewrite

$$\lambda_i^a(x)\lambda_i^a(x) = -(1/g^2)\mathrm{tr}(\tau^a U^\dagger \partial_i U)\mathrm{tr}(\tau^a U^\dagger \partial_i U) = -(2/g^2)\mathrm{tr}(U^\dagger \partial_i U U^\dagger \partial_i U) \ . \quad (3.13)$$

In this low momentum approximation, we also neglected the space dependence of $S_{ij}^{ab}(x)$ in the term SGS^T in (3.4); then, using the fact that S is an orthogonal matrix [Eq. (3.8)], one gets $SGS^T \to G$.

Strictly speaking, because of the Z_N local symmetry of the original theory (3.4), the action for the low momentum modes is slightly different. The derivatives should be understood as Z_N covariant derivatives. The most convenient way to write this action would be to understand $U(x)$ as belonging to U(N) rather than SU(N) and introduce a U(1) gauge field by

$$\Gamma_L = \frac{1}{2}\frac{M}{g^2}\mathrm{tr}\int d^3x(\partial_i - iA_i)U^\dagger(x)(\partial_i + iA_i)U(x) \ . \quad (3.14)$$

This defines a σ model on the target space U(N)/U(1), which is isomorphic to SU(N)/Z_N. This action does not look too bad. Even though it still cannot be solved exactly, it is amenable to analysis by standard methods, such as the mean field approximation, which in three dimensions and for large number of fields should give reliable results.

The suggestion, therefore, is the following. Let us integrate perturbatively the high momentum modes of the field $U(x)$. This is the renormalization group (RG) transformation. We would like to integrate out all modes with momenta $k^2 > M^2$. This procedure will necessarily generate a *local* effective action for the low momentum modes. At the same time, because of the (presumable) equivalence of the RG flows in QCD and our effective σ model, the effective coupling constant will be the running QCD coupling constant $\alpha_{\mathrm{QCD}}(M)$ at a scale M. This part of the theory can then be solved in the mean field approximation. Clearly, in order for the perturbative RG transformation to be justified, the QCD running coupling constant at the scale M must be small enough. Our procedure will then make sense, provided the energy will be minimized at the value of the variational parameter, for which

$$\alpha_{\mathrm{QCD}}(M) < 1 \ . \quad (3.15)$$

We will check whether this consistency condition is satisfied at the end of the calculation. In the next section we will calculate the expectation value of the Hamiltonian in the lowest order of this approximation scheme and perform the minimization with respect to M.

Before doing that, we would like to make one side remark. It is amusing to see how the present framework can accommodate instanton effects. Recall that in a path integral formalism instantons describe the tunneling transition between some initial state $\Phi[A]$ and a new state $\Phi[\tilde{A}]$ where a field $\tilde{A}_i$ is obtained from a field A_i by a large gauge transformation which is described by a nontrivial element of the homotopy group $\mathrm{II}_3(\mathrm{SU}(N)/Z_N) = Z$. The target space of the effective σ model, SU(N/Z_N, has the right topology: $\mathrm{II}_3(\mathrm{SU}(N)/Z_N) = Z$. The model therefore must have classical "hedgehog" solutions analogous to Skyrmions [11]. In fact, in the perturbative regime they should be easy to find. At weak coupling the action reduces to (up to a numerical coefficient) $\int d^3x\, d^3y\, \mathrm{tr}\left[U^\dagger(x)\partial_i U(x)\frac{1}{(x-y)^4}U^\dagger(y)\partial_i U(y)\right]$, and the equation of motion for $U(x)$ becomes relatively simple. Note also that this action has a dilatational invariance $x \to \lambda x$, $U(x) \to U(\lambda x)$ and also that the Skyrmion solution must approach 1 asymptotically at large distances. These functions $U_{\mathrm{cl}}(x)$ then correspond to contributions of the off-diagonal matrix elements between the initial Gaussian and the same Gaussian gauge transformed by a large gauge transformation, which is precisely the meaning of one instanton contribution to the path integral.

Note that the dilatational invariance is broken in our ansatz for slowly varying modes, by the appearance of the scale M. Indeed, the only Skyrmion solutions in the low momentum effective action (3.14) are pointlike, as a result of Derrick's collapse. This means, physically, that the scale M sets the nonperturbative infrared limit on the instanton size. Our variational vacuum, therefore, is free from the infrared problem associated with the large size instantons.

The variational ansatz which has been considered corresponds to a zero value of the QCD θ parameter, since we have integrated over the entire gauge group without any extra phases. As is well known, the general θ vacuum is defined as

$$|\theta\rangle = \sum_n e^{in\theta}|n\rangle \ , \quad (3.16)$$

where n labels the topological sectors in the configuration space [space of all potentials $A_i^a(x)$]. Generalization of our trial wave functions to nonzero θ is trivial; all we need to do is to insert in Eq. (2.12) an extra phase factor in the integrand:

$$\exp\left\{i\frac{\theta}{24\pi^2}\int dx\,\epsilon_{ijk}\mathrm{tr}\left[(U^\dagger\partial_i U)(U^\dagger\partial_j U)(U^\dagger\partial_k U)\right]\right\} \ . \quad (3.17)$$

The integrand here is a properly normalized topological charge, and it takes integer values for topologically nontrivial configurations $U(x)$; i.e., this factor reproduces the $\exp(in\theta)$ term in (3.16). This phase factor can be obtained also if one remembers that usually the θ dependence of the wave functional is given by the $\exp[i\theta S_{\mathrm{CS}}(A)]$, where $S_{\mathrm{CS}}(A)$ is a Chern-Simons term, which under the gauge transformation U transforms as

$$S_{\mathrm{CS}}(A^U) = S_{\mathrm{CS}}(A) + \frac{1}{24\pi^2}dx\,\epsilon_{ijk}\mathrm{tr}\left[U^\dagger\partial_i U)(U^\dagger\partial_j U)(U^\dagger\partial_k U)\right] ; \quad (3.18)$$

thus, integrating over U leads precisely to the phase fac-

tor (3.17). The state thus constructed is an eigenstate of an operator of the large gauge transformation with eigenvalue $e^{i\theta}$. This will result in addition of the same topological term to the effective action (3.4). It is amusing that for $\theta = \pi$ the "Skyrmions" in the effective theory will be "fermions." In the rest of this paper, we shall ignore instanton contributions, but it will be interesting to come back to this question later.

IV. SOLVING THE VARIATIONAL EQUATION

We will now calculate the expectation value of the energy:

$$H = \tfrac{1}{2}\int d^3x\, E_i^{a2} + \tfrac{1}{2}\int d^3x(\epsilon_{ijk}\partial_j A_k^a)^2 + \tfrac{1}{2}g\epsilon_{ijk}\epsilon_{ilm}f^{abc}\int d^3x\,\partial_j A_k^a A_l^b A_m^c + \frac{g^2}{8}\epsilon_{ijk}\epsilon_{ilm}f^{abc}f^{ade}\int d^3x\, A_j^b A_k^c A_l^d A_m^e \ . \tag{4.1}$$

We first perform the Gaussian integrations over the vector potential at fixed $U(x)$. Let us consider, for example, the calculation of the chromoelectric energy:

$$\int d^3x\langle E_i^{a2}\rangle_A = \int d^3x\Big\langle -\frac{\delta}{\delta A_i^a(x)}\frac{\delta}{\delta A_i^a(x)}\Big\rangle_A = TrG^{-1} - \int d^3x d^3y d^3z G^{-1}(x-y)G^{-1}(x-z)\langle A_i^a(y)A_i^a(y)(z)\rangle_A \ . \tag{4.2}$$

Using (3.4), it is easy to calculate the average over A. Defining, for convenience,

$$a_i^a(x) = \int d^3y\, d^3z\, \lambda_i^b(y)G^{-1}(y-z)S^{abc}(z)(\mathcal{M}^{-1})^{ca}(z,x) \ , \tag{4.3}$$

so that Gaussian integration over A is $\int DA\exp[-\frac{1}{2}(A+a)\mathcal{M}(A+a)]$, one gets

$$\int d^3x\langle E_i^{a\ 2}\rangle_A = 3(N^2-1)\int d^3x\, G^{-1}(x,x) - \int d^3x(G^{-1}\mathcal{M}^{-1}G^{-1})_{ii}^{aa}(x,x) - \int d^3x\, d^3y\, a_i^a(x)G^{-2}(x-y)a_i^a(y) \ , \tag{4.4}$$

where $G^{-2}(x-y) = \int d^3z\, G^{-1}(x-z)G^{-1}(z-y)$ and $\mathcal{M}^{-1}$ is defined as $\int d^3y\,\mathcal{M}^{-1}(x,y)\mathcal{M}(y,z) = \delta^3(x-z)$. Let us note that G^{-2} has dimension $[x]^{-5}$ and $\mathcal{M}^{-1}$ has dimension $[x]^{-2}$. For chromomagnetic field the calculations are straightforward and one gets

$$\langle(\epsilon_{ijk}\partial_j A_k^a)^2\rangle_A = (\epsilon_{ijk}\partial_j a_k^a)^2 + \epsilon_{ijk}\epsilon_{ilm}\partial_i^x\partial_l^y(\mathcal{M}^{-1})_{km}^{aa}(x,y)|_{x=y} \ , \tag{4.5}$$

$$\langle\partial_j A_k^a A_l^b A_m^c\rangle_{\hat{A}} = \partial_j a_k^a a_l^b a_m^c + \partial_j a_k^a(\mathcal{M}^{-1})_{lm}^{bc}(x,x) + a_l^b\partial_j^x(\mathcal{M}^{-1})_{km}^{ac}(x,y)|_{x=y} + a_m^c\partial_j^x(\mathcal{M}^{-1})_{kl}^{ab}(x,y)|_{x=y} \ , \tag{4.6}$$

$$\epsilon_{ijk}\epsilon_{ilm}f^{abc}f^{ade}\langle A_j^b A_k^c A_l^d A_m^e\rangle_A = 2f^{abc}f^{ade}a_j^b a_k^c a_l^d a_m^e + 8f^{abc}f^{ade}a_i^b a_i^d(\mathcal{M}^{-1})^{ce}(x,x) + 12f^{abc}f^{ade}(\mathcal{M}^{-1})^{bd}(x,x)(\mathcal{M}^{-1})^{ce}(x,x) \ . \tag{4.7}$$

Here we have used the obvious notation $\mathcal{M}_{ij}^{ab} = \mathcal{M}_{ab}\delta_{ij}$. The next step is to decompose the matrix field $U(x)$ into low and high momentum modes. In general, this is a nontrivial problem. However, since we are only going to integrate over the high momenta in the lowest order in perturbation theory, for the purposes of our calculation we can write

$$U(x) = U_L(x)U_H(x) \ , \tag{4.8}$$

where U_L contains only modes with momenta $k^2 < M^2$, U_H has the form $U_H = 1 + ig\tau^a\phi_H^a$, and ϕ_H contains only momenta $k^2 > M^2$. This decomposition is convenient, since it preserves the group structure. Also, since the measure DU is group invariant, we can write it as $DU_L DU_H$. With this decomposition we have

$$\lambda_i^a(x) = S_H^{ab}(x)\lambda_{iL}^b(x) + \lambda_{iH}^a(x) \ . \tag{4.9}$$

Further simplifications arise, since we only have to keep the leading piece in ϕ_H^a. We can therefore write in our approximation

$$
\begin{aligned}
S^{ab}(x) &= S_L^{ab}(x) \ , \\
\mathcal{M}^{ab}(x,y) &= 2\delta^{ab}G^{-1}(x-y) \ , \\
\lambda_i^a(x) &= \lambda_{iL}^a(x) + \lambda_{iH}^a(x) \ , \\
a_i^a(x) &= \tfrac{1}{2}\lambda_{iL}^a(x) + \tfrac{1}{2}\lambda_{iH}^b(x)S_L^{ba}(x) \ .
\end{aligned} \tag{4.10}
$$

We are now in the position to rewrite different pieces in the VEV of energy in this approximation:

$$
\begin{aligned}
\int d^3x \langle {E_i^a}^2 \rangle_A = \frac{3(N^2-1)}{2} \int G^{-1}(x,x) \\
-\tfrac{1}{4}\int d^3x\, d^3y \lambda_{iL}^a(x) G^{-2}(x-y)\lambda_{iL}^a(y) - \tfrac{1}{4}\int d^3x\, d^3y \lambda_{iH}^a(x) G^{-2}(x-y)\lambda_{iH}^a(y) \ .
\end{aligned} \tag{4.11}
$$

The cross term vanishes, since to this order, as we shall see, there is a decoupling between the high and the low momentum modes in the action, and therefore the product factorizes, and $\langle \lambda_{iH}^a \rangle = 0$. Our ansatz for G^{-1} [Eq. (2.16)] allows us to simplify this expression further. Remember that $\lambda_L(x)$ contains only momenta below M. Then it is immediate to see that

$$
\int d^3x\, d^3y\, \lambda_{iL}^a G^{-2}(x-y)\lambda_{iL}^a(y) = M^2 \int dx\, \lambda_{iL}^a(x)\lambda_{iL}^a(x) \ . \tag{4.12}
$$

We can then rewrite Eq. (4.11) as

$$
\int d^3x \langle {E_i^a}^2 \rangle_A = \frac{3(N^2-1)}{2} \int G^{-1}(x,x) - \frac{M^2}{4}\int d^3x\, \lambda_{iL}^a(x)\lambda_{iL}^a(x) - \tfrac{1}{4}\int d^3x\, d^3y\, \lambda_{iH}^a(x) G^{-2}(x-y)\lambda_{iH}^a(y) \ . \tag{4.13}
$$

The contribution of the magnetic term to the energy is very simple. All cross terms between the low and high momentum modes drop out. Some vanish for the same reason as the cross terms in Eq. (4.11) and others because they are explicitly multiplied by a power of the coupling constant. Since our approximation is the lowest order in g, except for the nonanalytic contributions that come from the low mode effective action, those terms do not contribute. In fact, the entire low momentum mode contribution drops out of this term. The reason is that the only terms which could give a leading order contribution is

$$
\int (\epsilon_{ijk}\partial_j \lambda_{kL}^a)^2 \ . \tag{4.14}
$$

It can be rewritten as

$$
(f_{ijL}^a)^2 + O(g^2) \ , \tag{4.15}
$$

where f_{ijL}^a is the "magnetic field" corresponding to the "vector potential" λ_{iL}^a. However, λ_L has the form of a pure gauge vector potential. Therefore $f_{ijL}^a = 0$ and the contribution of this term is higher order in g^2. We have checked that including this term indeed changes the energy density in the best variational state by a small amount ($\sim 10\%$), but has no effect at all on the best value of the variational parameter M. The entire magnetic field contribution to the energy is then

$$
\tfrac{1}{2}\langle B^2 \rangle_A = \tfrac{1}{8}(\epsilon_{ijk}\partial_j \lambda_{kH}^a)^2 + \frac{N^2-1}{2}\partial_i^x \partial_i^y G(x-y)|_{x=y} \ . \tag{4.16}
$$

The last step is to perform an averaging over the U field. For convenience, we rewrite here the complete expression for the energy density (here $V = \int d^3x$ is a space volume)

$$
\begin{aligned}
\frac{\langle 2H \rangle}{V} = \frac{3(N^2-1)}{2} G^{-1}(x,x) + (N^2-1)\partial_i^x \partial_i^y G(x-y)|_{x=y} \\
-\frac{1}{4V}\int d^3x\, d^3y \langle \lambda_{iH}^a(x) G^{-2}(x-y)\lambda_{iH}^a(y)\rangle_U + \tfrac{1}{4}\langle (\epsilon_{ijk}\partial_j \lambda_{kH}^a)^2 \rangle_U - \frac{M^2}{4V}\int d^3x \langle \lambda_{iL}^a(x)\lambda_{iL}^a(x)\rangle_U \ ,
\end{aligned} \tag{4.17}
$$

where the averaging over the U field should be performed with the σ model action (3.4). In our approximation this action has a simple form. Using Eq (4.10), we obtain

$$
\Gamma = \tfrac{1}{4}\int dx\, dy\, \lambda_{iH}^a(x) G^{-1}(x-y)\lambda_{iH}^a(y) + \frac{M}{4}\int dx\, \lambda_{iL}^a(x)\lambda_{iL}^a(x) \ . \tag{4.18}
$$

The low momentum mode part is precisely equal to Γ_L in Eq. (3.14). The only difference is that the coupling constant

that appears in this action should be understood as the running coupling constant at the scale M. This obviously is the only order 0 effect of the high momentum modes on the low momentum effective action:

$$\Gamma_L = \frac{1}{2}\frac{M}{g^2(M)}\mathrm{tr}\int d^3x(\partial_i - iA_i)U^\dagger(x)(\partial_i + iA_i)U(x) \ . \tag{4.19}$$

We are now in a position to evaluate the VEV of the energy. The contribution of the high momentum modes is immediately calculable. Using the parameterization $U_H(x) = 1 - \frac{i}{2}g\phi^a\tau^a$, we find that ϕ^a are free fields with the propagator

$$\langle\phi^a(x)\phi^b(y)\rangle = 2\delta^{ab}[\partial_i^x\partial_i^y G^{-1}(x-y)]^{-1}|_{p^2>M^2} \ . \tag{4.20}$$

Also to this order $\lambda^a_{iH}(x) = \partial_i\phi^a(x)$ and therefore $\epsilon_{ijk}\partial_j\lambda^a_{kH} = 0$. Using (4.20), one can see that

$$\tfrac{1}{4}\int d^3x\, d^3y\langle\lambda^a_{iH}(x)G^{-2}(x-y)\lambda^a_{iH}(y)\rangle_U = V\frac{N^2-1}{2}\int_M^\Lambda \frac{d^3k}{(2\pi)^3}G^{-1}(k) \ , \tag{4.21}$$

where Λ is the ultraviolet cutoff and the contribution of the high momentum modes to the energy [first two lines in Eq. (4.17)] is

$$\begin{aligned}\frac{2E_0}{V} &= (N^2-1)\left\{\int_0^\Lambda \frac{d^3k}{(2\pi)^3}\left[G^{-1}(k) + k^2G(k)\right] + \tfrac{1}{2}\int_0^M \frac{d^3k}{(2\pi)^3}G^1(k)\right\} \\ &= \frac{N^2-1}{2\pi^2}\left\{\int_0^M k^2dk\left[\frac{3}{2}M + \frac{k^2}{M}\right] + 2\int_M^\Lambda k^3dk\right\} = \frac{N^2-1}{10\pi^2}M^4 + \cdots \ .\end{aligned} \tag{4.22}$$

Terms denoted by the ellipsis in Eq. (4.22) depend on Λ, but are independent of the variational scale[2] M. We now have to evaluate the contribution of the low momentum modes. It is clear from the form of the action (4.19) that this contribution as a function of M will not be featureless. The most convenient way to think about it is from the point of view of classical statistical mechanics. Comparing Eqs. (4.17) and (4.19), we see that we have to evaluate the internal energy of the σ model (with the UV cutoff M) at the temperature proportional to the running coupling constant $g^2(M)$. For large[3] M, the coupling constant is small, which corresponds to the low temperature regime of the σ model. In this regime the global SU(N) $\otimes$ SU(N) symmetry group of the model is spontaneously broken. Lowering M, we raise $g^2(M)$ and therefore the temperature. At some critical value g_C, the model will undergo a phase transition into the unbroken (disordered) phase. Clearly, in the vicinity of the phase transition all thermodynamical quantities will vary rapidly, and therefore this is a potentially interesting region of coupling constants.

Before analyzing the phase transition region, let us calculate $E(M)$ for large M. In this regime the low momentum theory is weakly coupled. The calculation is straightforward and to lowest order in g^2 gives

$$\tfrac{1}{4}M^2\langle\lambda^a_{iL}(x)\lambda^a_{iL}(x)\rangle = \frac{N^2-1}{12\pi^2}M^4 \ . \tag{4.23}$$

Putting this together with the high momentum contribution, we find

$$\frac{E(M)}{V} = \frac{N^2-1}{120\pi^2}M^4, \quad M \gg \Lambda_{\mathrm{QCD}} \ . \tag{4.24}$$

This indeed is the expected result. The energy density monotonically increases as M^4, with the slope which is given by the standard perturbative expression. Note, however, that the slope is very small and the contribution of the low momentum modes to the energy is negative. Therefore, if the internal energy of the σ model grows significantly in the phase transition region, the sign of $E(M)$ could be reversed[4] and the energy will then be minimized for M in this region.

To see whether this indeed happens, we will now study the low momentum σ model in the mean field approximation. We rewrite the partition function by introducing a (Hermitian matrix) auxiliary field σ which imposes a unitarity constraint on $U(x)$:

[2]At this point we should note that our choice of the UV asymptotics of G is entirely consistent in the framework of our calculation, even without appealing to asymptotic freedom. In the calculation in Eq. (4.22) we could have taken $G(k) = \alpha k^\beta$. Minimization of the cutoff-dependent terms in (4.22) with respect to α and β would then give us precisely the form (2.16) at large k.

[3]Large M, of course, means large relative to Λ_{QCD}.

[4]The energy, of course, never becomes negative, since Eq. (4.22) contains a divergent M-independent piece. Here we concentrate only on the M dependence of E.

$$Z = \int DU\, D\sigma\, DA_i \exp(-\Gamma[U,A,\sigma]) \ ,$$

$$\Gamma[U,A,\sigma] = \frac{M}{2g^2(M)} \mathrm{tr} \int d^3x \left[(\partial_i - iA_i)U^\dagger(x)(\partial_i + iA_i)U(x) + \sigma \left(U^\dagger U - 1 \right) \right] \ . \tag{4.25}$$

The role of the vector field A_i is to impose a U(1) gauge invariance and thereby to eliminate one degree of freedom. As far as the thermodynamical properties are concerned, its effect is only felt as an $O(1/N^2)$ correction. At the level of accuracy of the mean field approximation, we can safely disregard it, which we do in the following. The mean field equations are

$$\langle U^\dagger U \rangle = 1 \ , \tag{4.26}$$

$$\langle \sigma U \rangle = 0 \ . \tag{4.27}$$

From Eq. (4.27) it follows that either $\langle \sigma \rangle = 0$, $\langle U \rangle \neq 0$ (the ordered, broken symmetry phase with massless Goldstone bosons) or $\langle \sigma \rangle \neq 0$, $\langle U \rangle = 0$ (the disordered, unbroken phase with massive excitations). We are mostly interested in the disordered phase, since there the mean field approximation should be reliable. Since the symmetry is unbroken, the expectation value of σ should be proportional to a unit matrix

$$\langle \sigma_{\alpha\beta} \rangle = \sigma^2 1_{\alpha\beta} \ . \tag{4.28}$$

Equation (4.26) then becomes

$$2N^2 \frac{g^2(M)}{M} \int_0^M \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+\sigma^2} = \frac{N^2 g^2(M)}{\pi^2} \left(1 - \frac{\sigma}{M} \arctan \frac{M}{\sigma} \right) = N \ . \tag{4.29}$$

The gap equation (4.29) has solution only for couplings (temperatures) $g^2(M)$ larger than the critical coupling (temperature) g_C^2, which is determined by the condition that $\sigma = 0$:

$$\alpha_C = \frac{g_C^2}{4\pi} = \frac{\pi}{4} \frac{1}{N} \ . \tag{4.30}$$

The low momentum mode contribution to the ground state energy is

$$N^2 M \int_0^M \frac{d^3k}{(2\pi)^3} \frac{k^2}{k^2+\sigma^2} = \frac{N^2}{2\pi^2} M \left[\frac{1}{3} M^3 - \sigma^2 M + \sigma^3 \arctan \frac{M}{\sigma} \right] \ . \tag{4.31}$$

The final mean field expression for the ground state energy density is [we do not distinguish between N^2 and N^2-1 since we have neglected the contribution of the U(1) gauge field; the errors are of order $1/N^2$ and are definitely smaller than the error introduced by using the mean field approximation in the first place]

$$E = \frac{N^2}{4\pi^2} M^4 \left[-\frac{2}{15} + \frac{\sigma^2}{M^2} \frac{\alpha_C}{\alpha(M)} \right] \ , \tag{4.32}$$

where $\alpha(M)$ is the QCD coupling at the scale M, α_C is given by Eq. (4.30), and σ is determined by

$$\frac{\sigma}{M} \arctan \frac{M}{\sigma} = \frac{\alpha(M) - \alpha_C}{\alpha(M)} \ . \tag{4.33}$$

The energy as function of M is plotted in Fig. 1 for $N = 3$. Qualitatively, it is the same for any N. The minimum of the energy is obviously at the point $\alpha(M) = \alpha_C$. Using the one-loop Yang-Mills β function and $\Lambda_{\mathrm{QCD}} = 150$ MeV, we find, for $N = 3$,

$$M = \Lambda_{\mathrm{QCD}} e^{24/11} = 8.86 \Lambda_{\mathrm{QCD}} = 1.33 \text{ GeV} \ . \tag{4.34}$$

To see what is the phenomenological significance of this number, we have calculated the value of the gluon condensate $(\alpha/\pi)\langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle = (2\alpha/\pi)(\langle (B^a_i)^2 \rangle - \langle (E^a_i)^2 \rangle)$. After some calculations which are straightforward and do not contain any new ingredients, we get[5]

[5]We have again kept only the M-dependent pieces. Each one of the quantities $\langle (E^a_i)^2 \rangle$ and $\langle (B^a_i)^2 \rangle$ is of course positive, as a result of positive UV-divergent, but M-independent pieces. It is easy to check that the energy density $E = \frac{1}{2}(\langle (E^a_i)^2 \rangle + \langle (B^a_i)^2 \rangle) = -(1/30\pi^2) N^2 M^4$ coincides with the first term in Eq. (4.32), as it must.

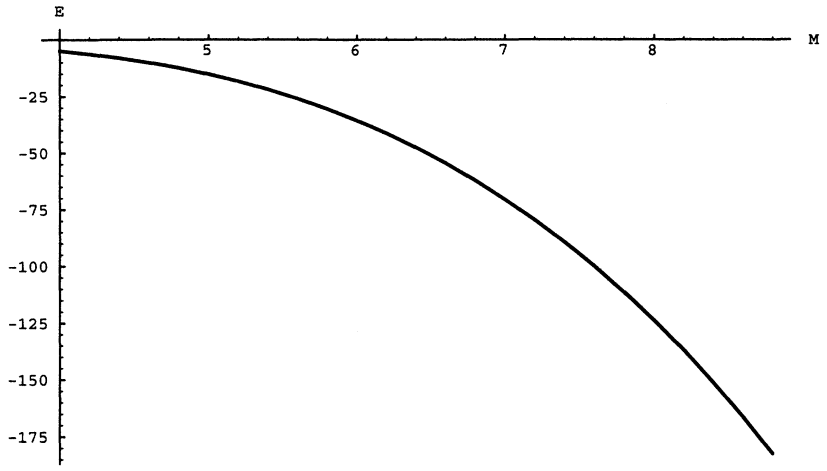

FIG. 1. Energy density of a variational state as a function of the variational parameter M in units of $\Lambda_{\rm QCD}$. The energy is only shown for $M < 8.86\Lambda_{\rm QCD}$, which corresponds to the disordered phase of the effective low momentum σ model. Close to the phase transition point in the ordered phase, the mean field approximation is not applicable. Far from the phase transition point, at large M the energy density is a monotonically increasing function of M given in Eq. (4.24).

$$\langle (E_i^a)^2 \rangle = -\frac{1}{24\pi^2} N^2 M^4, \quad \langle (B_i^a)^2 \rangle = -\frac{1}{40\pi^2} N^2 M^4 \ , \tag{4.35}$$

and finally obtain

$$\frac{\alpha}{\pi} \langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle = N M^4 \frac{1}{2} \left(-\frac{1}{40\pi^2} + \frac{1}{24\pi^2} \right) = \frac{N}{120\pi^2} M^4 = 0.008 \ \text{GeV}^4 \ . \tag{4.36}$$

The best phenomenological value of this condensate is 0.012 GeV^4 [5]. Considering that $\langle F^2 \rangle$ is proportional to the fourth power of M, our result is very reasonable. For example, changing M by only 10% from 1.33 to 1.46 GeV would give $(\alpha/\pi)\langle F^2 \rangle = 0.0116$ GeV^4, in perfect agreement with [5].

Note that for $N = 3$ the value of the QCD coupling constant at the variational scale is $\alpha_C = 0.26$. It is reasonably small, so that the consistency condition for the perturbative integration of the high momentum modes is satisfied. However, it is not so small that higher order corrections be negligible. We expect therefore that including higher orders in perturbation theory can give corrections to our result for $\alpha(M)$ of order 25%. Since M depends exponentially on $\alpha(M)$, such a change in α may change the value of M by a factor of 2–3. Consequently our result for F^2 should be taken only as an order-of-magnitude estimate. This is usually the case in theories with logarithmically running coupling constants. The best accuracy is always achieved for dimensionless quantities, since those usually are slowly varying functions of α. The overall scale depends on α exponentially and therefore always has the largest error. It would be interesting to calculate some dimensionless quantities, such as the ratio of the square of the string tension to the Shifman-Vainshtein-Zakharov (SVZ) condensate, in our approach [10].

Another uncertainty comes from the use of the mean field approximation. As a rule, the mean field approximation gives a good estimate of the critical temperature. Sometimes, however, it gives wrong predictions for the order of the phase transition. We believe that this is indeed the case here. Our results would indicate that the phase transition is second order. The mass gap in the σ model vanishes continuously at the critical point. The universality class describing the symmetry-breaking pattern $(\mathrm{SU}(N) \otimes \mathrm{SU}(N))/\mathrm{SU}(N)$ was considered in the context of finite temperature chiral phase transition in QCD. The results of ϵ expansion [12] and also numerical simulations [13] strongly suggest that the phase transition is of first order. In our case there is an additional Z_N symmetry in the game. However, if anything, we believe that its presence should increase the latent heat rather than turn the transition into a second order one. The reason is that the Z_N-gauge-invariant theory allows the existence of topological defects—the Z_N strings—and condensation of topological defects frequently leads to discontinuous phase transitions.

Nevertheless, we believe that our results are robust against this uncertainty. The mean field approximation should be reliable in the regime where the mass gap in the σ model is not too small. At the point $M = 4.5\Lambda_{\rm QCD}$, we find

$$\sigma = 0.23M, \quad \alpha(M) = 0.38 \ . \tag{4.37}$$

Since the gap is of the order of the UV cutoff, the mean field approximation is reliable in the vicinity of this point. The perturbation theory is also still reasonable at this value of α. The fact that the energy is negative and has a minimum for some $\alpha(M) < 0.38$ seems to be therefore unambiguous.

We now want to argue that, independently of the mean field calculation, it is physically very plausible that the energy is minimized precisely at the critical temperature on the disordered side of the phase transition (if

it is of the first order). Consider, first, the contribution of the high momentum modes to the ground state energy, Eq. (4.22). It is proportional to M^4 with a fixed (M-independent) proportionality coefficient $x = (N^2-1)/10\pi^2$. Consider now the low momentum contribution in the large M region, Eq. (4.23). It is again proportional to M^4 with the coefficient $y_0 = (N^2-1)12\pi^2$. The proportionality coefficient of the low momentum contribution at the phase transition point, according to our calculation, is twice as big, $y_C = 2N^2/12\pi^2$ (we disregard the difference between N^2 and N^2-1). This is very easy to understand physically. In the large M, low temperature regime the global symmetry of the σ model $SU(N)\otimes SU(N)$ is broken down spontaneously to $SU(N)$. This leads to the appearance of N^2-1 massless Goldstone bosons. In fact, at zero temperature, those are the only propagating degrees of freedom in the model. All the rest have masses of the order of the UV cutoff and therefore do not give any contribution to the internal energy. Now, when the temperature is raised (M is lowered), the Goldstone bosons remain massless and other excitations become lighter. If the transition is second order, at the phase transition point the symmetry is restored, one should have a complete multiplet of the $SU(N) \otimes SU(N)$ symmetry of massless particles. The dimensionality of this multiplet is $2(N^2-1)$. The contribution of every degree of freedom to the internal energy is still roughly the same as at zero temperature. This is so, since, although at the phase transition the particles are interacting, critical exponents of scalar theories in three dimensions are generally very close to their values in a free theory [14]. The internal energy at this point therefore should be roughly twice its value at zero temperature. Moving now to higher temperatures, all the particles become heavier, and therefore their contribution to internal energy decreases. The internal energy therefore should have a maximum at the phase transition temperature.

Note that the ground state energy of the Yang-Mills theory is the difference between the high momentum contributions and the internal energy of the low mode σ model. Already at zero temperature, these two contributions differ only by 20%, and that is why the coefficient in the expression equation (4.24), even though positive, is so small. At the critical point, where the low momentum mode internal energy is twice as large, the chances of the slope becoming negative are very good. This is indeed precisely what happens in our mean field analysis, but according to the previous argument this in large measure is independent of the approximation. If the phase transition is first order, one should be more careful. The internal energy then changes discontinuously across the phase transition. The particles in the disordered phase are always massive, and the internal energy is smaller than in the case of the second order phase transition. However, if the transition is only weakly first order, the same argument still goes through (the fact that the mean field predicts second order phase transition may be an indication that if it is in fact first order it is only weakly so). In fact, it does seem very likely that the ground state energy will become negative, since all is needed for that is that the σ model internal energy grow by 20% at the phase transition relative to the zero temperature limit. Moreover, in this case there will be a finite latent heat, which means that the internal energy in the disordered (high temperature) phase is higher. The ground state energy, therefore, will have its minimum in the disordered phase.

We believe, therefore, that our results are qualitatively correct and will survive the improvement of the approximation.

V. WILSON LOOP AND AREA LAW

The next interesting question is whether the variational state we found describes the physics of confinement. The relevant quantity to calculate is the Wilson loop

$$W(C) = \left\langle \mathrm{tr} P \exp\left(i\frac{g}{2} \oint_C dx_i A_i^a \tau^a \right) \right\rangle . \tag{5.1}$$

When averaging over A we must take into account the P ordering of the exponent—the simplest way to do it is to introduce new degrees of freedom living on the contour C which, after quantization, become the $SU(N)$ matrices τ^a [15]. We shall consider here how it works in the case of the SU(2) group—the generalization of this construction to an arbitrary Lie group has been discussed in [15].

The construction is based on the observation made in [16] that instead of considering the ordered product of τ^a matrices one can consider the correlation function

$$\left\langle \frac{\tau^a(t_1)}{2} \frac{\tau^b(t_2)}{2} \cdots \frac{\tau^c(t_k)}{2} \right\rangle \to \langle n^a(t_1) n^b(t_2) \cdots n^c(t_k) \rangle$$

$$= \int Dn(t) n^a(t_1) n^b(t_2) \cdots n^c(t_k) \exp\left[i(S+\tfrac{1}{2}) \int_\Sigma d^2\xi\, \epsilon_{\mu\nu} \epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c \right] , \tag{5.2}$$

where S is the spin of representation; i.e., for the fundamental representation $S = \frac{1}{2}$, $n^a(t)$ is a unit vector $n^a n^a = 1$ living on a contour C (t is a coordinate on the contour) and Σ is an arbitrary two-dimensional surface with the boundary $C = \delta\Sigma$. The two-dimensional action (here and later we shall concentrate only on case $S = \frac{1}{2}$)

$$S[n] = \int_\Sigma d^2\xi\, \epsilon_{\mu\nu} \epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c \tag{5.3}$$

depends only on values $n^a(t)$ at the boundary. The variation of the action is

$$\delta S = \oint_C dt\, \epsilon^{abc} n^a \partial_t n^b \delta n^c \ . \tag{5.4}$$

Here we have used the fact that $\delta n^a n^a = 0$ (because $n^c n^c = 1$) and thus $\epsilon_{\mu\nu}\epsilon^{abc}\partial_\mu n^a \partial_\nu n^b \delta n^c = 0$. It is easy to see that $\langle n^a(t_1) n^b(t_2) \cdots n^c(t_k)\rangle$ depends only on the ordering of $t_1, \ldots, t_k$, as it should. To see this and the fact that $n^a(t)$ behaves effectively as τ^a, let us make local field reparametrization

$$n^a(t) \to n^a(t) + \epsilon^{abc}\Omega^b(t) n^c(t) \ , \tag{5.5}$$

under which the action variation is $\delta S = -\oint_C dt\, \dot{n}^a(t)\Omega^a(t)$ and one gets the Ward identities (it is important to remember here that correlators in any QFT are averages of the T-ordered products)

$$\frac{d}{dt}\langle n^a(t) n^b(t_1)\cdots n^c(t_k)\rangle = i\sum_{i=1}^{k} \delta(t-t_i)\epsilon^{adf}\langle n^f(t) n^b(t_1)\cdots \underline{n^d(t_i)} \cdots n^c(t_k)\rangle \ , \tag{5.6}$$

where $\underline{n^d(t_i)}$ means the exclusion of this term from the products of the fields in a correlator. From (5.6) one can conclude immediately that correlation function indeed depends only on ordering of $t_1, \ldots, t_k$ and the following equal time commutation relations hold:

$$\left[n^a, n^b\right] = i\epsilon^{abc} n^c \ . \tag{5.7}$$

As a result, one can represent the Wilson loop [Eq. (5.1)] in the form

$$W(C) = \left\langle \int Dn(t) \exp\left[i \int_\Sigma d^2\zeta \epsilon_{\mu\nu}\epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c \right] \exp\left(ig \oint_C dx_i A_i^a(x(t)) n^a(t) \right) \right\rangle \tag{5.8}$$

and now we can average over A_i using (3.4) and (4.3),

$$\left\langle \exp\left(ig \oint_C dx_i A_i^a(x(t)) n^a(t) \right) \right\rangle_A = \exp\left(-ig \oint_C dx_i a_i^a(x(t)) n^a(t) \right)$$
$$\times \exp\left(-\tfrac{1}{2} \oint_C \oint_C dt_1 dt_2 \dot{x}_i(t_1)\dot{y}_i(t_2) n^a(t_1) n^b(t_2) (\mathcal{M}^{-1})^{ab}(x,y) \right) \ , \tag{5.9}$$

where a_i^a was defined in (4.3). Now the Wilson loop can be calculated as the average over two scalar fields, $U(x)$ living in the whole space and $n^a(\xi)$ living on a two-dimensional surface Σ, such that $C = \delta\Sigma$:

$$W(C) = \int DU \int Dn \exp(-\Gamma[U] + iS[n]) \exp\left(-ig \oint_C dx_i a_i^a(x(t)) n^a(t) \right)$$
$$\times \exp\left(-\tfrac{1}{2} \oint_C \oint_C dt_1 dt_2 \dot{x}_i(t_1)\dot{y}_i(t_2) n^a(t_1) n^b(t_2) (\mathcal{M}^{-1})^{ab}(x,y) \right) \ . \tag{5.10}$$

In the infrared limit which is of main interest to us here, we can use (4.10) to simplify (5.10) and get

$$W(C) = \int DU_L \int Dn \exp(-\Gamma_L[U] + iS[n]) \exp\left(-i\frac{g}{2} \oint_C dx_i \lambda_i^a{}_L(x(t)) n^a(t) \right)$$
$$\times \exp\left(-\tfrac{1}{4} \oint_C \oint_C dt_1 dt_2 \dot{x}_i(t_1)\dot{y}_i(t_2) n^a(t_1) n^a(t_2) G(x-y) \right)$$
$$\times \int DU_H \exp(-\Gamma_H[U]) \exp\left(-i\frac{g}{2} \oint_C dx_i \lambda_i^b H(x(t)) S_L^{ba} n^a(t) \right) \ . \tag{5.11}$$

Using Eq. (4.20), one can see that the last term in (5.11) after integrating over the U_H becomes equal to the second term and one gets finally

$$W(C) = \int Dn \exp(iS[n]) \exp\left(-\tfrac{1}{2} \oint_C \oint_C dt_1 dt_2 \dot{x}_i(t_1)\dot{y}_i(t_2) n^a(t_1) n^a(t_2) G(x-y) \right)$$
$$\times \int DU \exp(-\Gamma[U]) \exp\left[\tfrac{1}{2} \oint_C dx_i \mathrm{tr}(\tau^a U^\dagger \partial_i U) n^a(t) \right] \ , \tag{5.12}$$

where integrating DU is over low energy modes only and $\Gamma[U]$ is the corresponding low energy action. Since $G(x-y)$ is short range, the term

$$\exp\left(-\tfrac{1}{2}\oint_C\oint_C dt_1 dt_2 \dot{x}_i(t_1)\dot{y}_i(t_2)n^a(t_1)n^a(t_2)G(x-y)\right) \tag{5.13}$$

gives only perimeter dependence and one can neglect it when calculating the string tension. Then it can be shown, rewriting $n^a(t)$ as the τ^a and performing some simple algebra, that the calculation of the Wilson loop is closely related to the calculation of the vacuum expectation value of the monodromy operator

$$M = \mathrm{tr}P\exp\left(\oint_C dl_i U^\dagger\partial_i U\right) \tag{5.14}$$

in the low momentum σ model with an effective action [6] $\Gamma[U]$. Since the target space of the σ model is $\mathcal{M} = \mathrm{SU}(N)/Z_N$ and $\mathrm{II}_1(\mathcal{M}) = Z_N$, this factor can take on values $\exp(i2\pi n/N)$. It has a natural interpretation in terms of the topological defects in the σ model. As mentioned already, the topology allows existence of Z_N strings. The string creation operator and the operator M satisfy the commutation relations of the 't Hooft algebra [17]. Therefore, in the presence of a string, the operator M has expectation value $\exp(i2\pi n/N)$, where n is the linking number between the loop C and the string. As we have argued, the σ model is in the disordered phase. Usually, this means that the topological defects are condensed. The vacuum of the σ model must have therefore a large number of strings, and the VEV of $\mathcal{M}$, probably, will average to zero very quickly and for large loops will have an area law $W(C) \sim \exp(-\alpha' A)$. Strictly speaking, for this to happen one needs not only a large number of strings, but also a large fluctuation in this number, but those usually come together. Even though this scenario of the appearance of the area law is appealing, one should be aware of the fact that it is susceptible to the same criticism as any other approach relying on the existence of the nontrivial center of the gauge group. It does not provide a natural explanation of why adjoint Wilson loops in pure glue theory seem to follow approximately an area law up to some relatively large size or why this area law becomes exact in the limit of infinite N [18]. To answer these kinds of questions in the present framework would require a much more thorough study of the dynamics of the effective σ model.

We also would like to mention that the model of two fields U and n^a defined in (5.12) is of some interest in itself. For example, one can study how nonperturbative fluctuations of both fields, Z_N strings and Skyrmions for U and instantons for n, interact with each other. These questions as well as a calculation of α' will be considered in [10].

An interesting point is that if one couples fundamental fermions to the Yang-Mills fields the effective σ model will not have a Z_N gauge symmetry any more. The origin of this Z_N symmetry is the fact that the Yang Mills fields do not transform under the center of the gauge group. Fundamental fermions, however, do transform nontrivially, and therefore the σ model action will depend on these matrices U. The target space now therefore is $\mathrm{SU}(N)$, rather than $\mathrm{SU}(N)/Z_N$, and is simply connected. The topology of the target space does not allow strings any more. Therefore, if it is true that it is the condensation of these objects that is responsible for the area law for the Wilson loop, the area law will disappear. This is in complete agreement with one's expectations, that in a theory with fundamental charges an external test charge can be screened, and therefore there is no area law for the Wilson loop.

VI. DISCUSSION AND CONCLUSION

In this paper we have presented a simple variational calculation of the Yang-Mills ground state WF. Our trial states preserved gauge invariance explicitly. The results are encouraging. We find that the energy is minimal for a state which is different from the perturbative vacuum, even though the perturbative vacuum state was included in our variational ansatz. Dynamical scale generation takes place, and the gluon (SVZ) condensate in the best variational state is nonzero.

It is interesting to note that from the point of view of the effective σ model the energy is minimized in the disordered (unbroken) phase. In other words, the fluctuations of the field U are big, unlike in the perturbative regime (high momentum modes), where U is very close to a unit matrix. From the point of view of the original WF, this means that the off-diagonal contributions, coming from the Gaussian gauge rotated by a slowly varying gauge transformations, are large. This is telling us that it was indeed necessary to project the initial Gaussian onto a gauge-invariant state. Without doing this or doing this only perturbatively, we would miss the important contributions of the off-diagonal elements to the energy expectation value. This is perhaps the most important qualitative lesson to be learned from the present work. Taking these off-diagonal contributions into account nonperturbatively is the main distinction between our approach and previous work [7,9].

It is important to realize that even though our approach starts with a Gaussian wave functional which has

[6] In fact, the Wilson loop does not reduce to M, but rather to $\mathrm{tr}P\exp\left(\frac{1}{2}\oint_C dl_i U^\dagger\partial_i U\right)$. We believe, however, that qualitatively its behavior should be similar.

a zero expectation value of magnetic field, it is not necessarily unrelated with other approaches to QCD vacuum, which focus on the presence of nonzero magnetic field in the ground state [9,19]. A Gaussian wave function with the width G^{-1}, different from the perturbative one, can be always decomposed in a trivial way into a superposition of coherent states with the perturbative width G_0^{-1} :

$$\exp\{-\tfrac{1}{2}AG^{-1}A\}$$

$$\sim \int Da \exp\{-\tfrac{1}{2}AG_0^{-1}A + aA - \tfrac{1}{2}aKa\} \ , \quad (6.1)$$

with $K^{-1} = G_0^{-1} - G^{-1}$. A state $\exp\{-\frac{1}{2}AG_0^{-1}A + aA\}$ does have a nonzero expectation value of the magnetic field $B_i \sim \epsilon_{ijk}\partial_j a_k$. One can therefore think about our variational state as a gauge-invariant quantum superposition of states with nonzero magnetic field. Of course, superposing states with different magnetic field is also a necessary final step in the picture of the Copenhagen (or spaghetti) vacuum [19], which serves to restore gauge, rotational, and translational invariance of the vacuum state.

There is still a lot of work to be done, even in the framework of our variational ansatz. Our present paper should be considered only as an exploratory research. Of course, coupling the fermions is a very interesting question in itself. It seems to us that it should be possible to treat a theory with fermions in basically the same variational approach as presented here. It would be then very interesting to see the chiral symmetry breaking and calculate fermionic condensates.

Quite apart from this, there are several technical points that can be improved. First, we are planning to extend the RG calculation to take into account the one-loop contribution of the high momentum modes. This might require to change a variational ansatz slightly. One may have to consider not the gauge-projected Gaussians, but gauge-projected products of Gaussians and polynomials of the fourth order in the fields. This does not change the level of complexity of the calculation.

It would also be desirable to have better methods to deal with the low momentum σ model, especially since we suspect that the mean field approximation does not give the correct order of the phase transition. Although we do not expect the variational parameter to be very sensitive to this, the vacuum condensates can depend strongly on the mass gap of the σ model.

Finally, there is one more direction in which the calculation can be extended. In this paper we have adopted the simplest ansatz for the width of the Gaussian G based on the argument that it should be short ranged. The Fourier transform of our propagator goes to a constant at zero momentum. This, however, is not the only possible form of a short range correlator. It could have a different small momentum behavior. It is quite possible that the small momentum behavior is very important. One could therefore introduce an additional variational parameter γ, assuming the asymptotic small momentum dependence of the function G to be of the form k^γ. This will only affect the last step of our calculation. The action of the effective low momentum model will have extra derivatives.

In conclusion, it seems to us that the type of the variational approximation presented here is manageable and also gives some preliminary interesting results. It therefore warrants further work along the lines described in this section.

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